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The ground state of chargeless fermions with finite magnetic moment

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Abstract

We consider the ground state of a system of chargeless fermions, such as neutrinos, of mass m and magnetic moment μ interacting through long-range magnetic dipole interaction, within the framework of a Hartree–Fock variational approach. At high densities the uniform paramagnetic state becomes unstable towards a ferromagnetic state with quadrupolar deformation of the Fermi surface. The exchange energy which is attractive dominates the repulsive kinetic energy. If we let the density be a variable, then above a certain density the system will collapse to an infinite density state unless another short-range interaction stops the collapse. In the case of large deformations, the possibility of a purely dipolar deformation exists.

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1. Introduction

In the context of big-bang cosmology, it has recently been suggested that the universal cosmic background neutrinos (the relic neutrinos) may be in a ferromagnetic state with domain walls between different regions [1]. Neutrinos are spin-1/2 fermions with no electric charge but a finite mass and a finite magnetic moment (Dirac neutrinos). Whether or not such neutrinos with a tiny magnetic moment could have condensed into a ferromagnetic state, the general problem of chargeless (neutral) fermions interacting via magnetic dipole interaction is a very interesting problem in its own right. Fundamental questions of interest are the nature of the ground state and low lying excitations. For definiteness, in this paper we consider the case of spin-1/2 Fermi particles of mass m, spin $\vec{s}$ and magnetic moment $\mu\vec{s}$, interacting through magnetic dipole interaction. We study the ground state of this system using the familiar many-body variational Hartree–Fock approach.

Unlike the case of spin-1/2 charged fermions interacting with spin-independent Coulomb interaction $e^2/|\vec{r}_1 - \vec{r}_2|$, the magnetic dipole interaction between two chargeless particles such as neutrinos (1 and 2) has the form of the non-central spin-dependent tensor interaction [2], $v(r)(\vec{s}_1 \cdot \vec{s}_2 - 3\vec{s}_1 \cdot \hat{r}\vec{s}_2 \cdot \hat{r})$, where $\vec{s}_i$ denotes the particle spin, $\vec{r} = \vec{r}_1 - \vec{r}_2$, $r = |\vec{r}_1 - \vec{r}_2|$, $\hat{r} = \vec{r}/r$, and where $v(r)$ in the present case falls of as μ^2/r^3 with respect to the inter-particle distance r. The interaction depends on the direction of $\vec{r}$ in an essential way, containing the five components of the spherical harmonics $Y_{2\mathrm{m}}(\theta, \phi) \equiv Y_{2\mathrm{m}}(\hat{r})$. In general, at very short average distances (high density) the interaction term varying as $1/r^3$ will dominate the system energy. This is just the opposite of the familiar electron gas case where the Coulomb interaction varying as $1/r$ dominates only at low densities [2]. For the latter, for average inter-particle separation r_0, the interaction energy goes as $1/r_0$ whereas the kinetic energy goes as $1\big/r_0^2$. For the case of the magnetic dipole interaction, if we use the dimensionless density parameter $r_{\mathrm{sm}} = r_0/r_{\mathrm{m}}$, where $(4\pi/3)r_0^3 = V/N = 1/n$ and $r_{\mathrm{m}} = 2m\mu^2/\hbar^2$ (magnetic radius), the kinetic energy of the system varies as $1\big/r_{\mathrm{sm}}^2$, whereas the interaction term varies as $1\big/r_{\mathrm{sm}}^3$.

It is important to note that in the Fourier transformed space (the $\boldsymbol{q}$-space), the spin-dependent dipole interaction is independent of the magnitude of $\boldsymbol{q}$, but depends on its direction $(\hat{q})$, containing a sum of terms of the form $\mu^2 M_{12}^{(M)} Y_{2,-M}(\hat{q})$, for $\vec{q} \neq 0$. For $\vec{q} = 0$, the Fourier component of the interaction vanishes identically. There is no longer an isotropic $4\pi e^2/q^2$ form as in the case of the Coulomb interaction. Here, $N_{12}^{(M)}$, $M = -2$ to +2, are two-particle spin operators which connect states whose z-components of the total spin, $S_Z = s_{1z} + s_{2z}$, differ by M. Since there is no $\vec{q} = 0$ term in the interaction, the Hartree term (the direct term) goes to zero in the uniform density case. The Hartree–Fock (HF) exchange contribution with an overall negative sign comes from only the $N_{12}^{(0)} Y_{20}(\hat{q})$ term in the interaction involving no change in the total S_Z. Both parallel spins and anti-parallel spins contribute equally, but with opposite signs. The exchange contribution will go to zero if the occupation number $n_\sigma(\vec{k})$ of the particles is independent of the spin index σ (i.e. if $n_\uparrow(\vec{k}) = n_\downarrow(\vec{k})$) or if $n_\sigma(\vec{k})$ does not depend on the direction of $\vec{k}$ (i.e. if $n_\sigma(\vec{k}) = n_\sigma(k)$). Thus, for a nonzero exchange contribution, one must have $n_\uparrow(\vec{k}) \neq n_\downarrow(\vec{k})$, i.e. a spin-polarized state, and one must have a deformed Fermi surface, with $n_\sigma(\vec{k}) = n_\sigma^{(0)}(k) + \delta n_\sigma(\vec{k})$, where the deformation $\delta n_\sigma(\vec{k})$ depends on the direction of $\vec{k}$. At high enough density ($r_{\mathrm{sm}} < 1$), we show that such a ground state $|\Psi_0\rangle$ becomes more stable than the state $|0\rangle$ corresponding to the unpolarized (paramagnetic) isotropic Fermi surface of the non-interacting gas. In terms of the parameters for the assumed deformation, we determine the value of r_{sm} at which this transition will take place. Beyond this density, there is of course no stopping, and the system would collapse, unless some other inter-particle interaction at very short distances stops the collapse.

The arrangement of the paper is as follows. In section 2, we calculate the HF energy of two particles interacting via magnetic dipole interaction, and set up the many-body Hamiltonian. In section 3, the many-body variational HF approximation is used to calculate and analyse the expression for the total energy $E = E_{\mathrm{kin}} + E_{\mathrm{exch}}$ of the system in the new variational ground state $|\Psi_0\rangle$, with arbitrary occupation number $n_\sigma(\vec{k})$. Our conclusions and possible effects of higher order corrections are discussed in section 4.

2. Two-particle Hartree–Fock energy and the many-body Hamiltonian

The magnetic dipole interaction between two particles of spin $\vec{s}$ and magnetic moment $\mu\vec{s}$ is given by

$$V(\vec{r}_1\vec{s}_1; \vec{r}_2\vec{s}_2) \equiv V(1,2) = \frac{\mu^2}{r^3}[\vec{s}_1 \cdot \vec{s}_2 - 3\vec{s}_1 \cdot \hat{r}\vec{s}_2 \cdot \hat{r}], \tag{2.1}$$

where

$$\vec{r} = \vec{r}_1 - \vec{r}_2, \qquad r = |\vec{r}_1 - \vec{r}_2|, \qquad \hat{r} = \vec{r}/r. \tag{2.2}$$

This interaction can be decomposed into the form [3]

$$V(1.2) = \frac{\mu^2}{r^3} \sum_{M=-2}^{+2} N_{12}^{(M)} F^{(-M)}(\widehat{r}), \qquad M = 0, \pm 1, \pm 2, \tag{2.3}$$

where the two-particle spin operators $N_{12}^{(M)}$ and the angular part $F^{(-M)}(\widehat{r})$ are given by

$$N_{12}^{(0)} = s_1^{(0)} s_2^{(0)} - \frac{1}{4}\left(s_1^{(+1)} s_2^{(-1)} + s_1^{(-1)} s_2^{(+1)}\right), \tag{2.4a}$$

$$N_{12}^{(\pm 1)} = s_1^{(\pm 1)} s_2^{(0)} + s_1^{(0)} s_2^{(\pm 1)}, \tag{2.4b}$$

$$N_{12}^{(\pm 2)} = s_1^{(\pm 1)} s_2^{(\pm 1)}, \tag{2.4c}$$

$$F^{(0)}(\widehat{r}) = -\sqrt{\frac{16\pi}{5}} Y_{20}(\widehat{r}), \tag{2.5a}$$

$$F^{(\mp 1)}(\widehat{r}) = \mp\sqrt{\frac{6\pi}{5}} Y_{2,\mp 1}(\widehat{r}), \tag{2.5b}$$

$$F^{(\mp 2)}(\widehat{r}) = -\sqrt{\frac{16\pi}{5}} Y_{2,\mp 2}(\widehat{r}). \tag{2.5c}$$

Here, $Y_{2m}(\widehat{r})$ denotes the spherical harmonics of order 2, and

$$s_i^{(0)} = s_{iz}, \qquad s_i^{(\pm 1)} = s_{ix} + i s_{iy}, \qquad i = 1, 2, \tag{2.6}$$

are the usual spin-$1/2$ operators.

A straightforward calculation of the Fourier transform (FT) of the interaction with respect to $\vec{r}$ leads to the expression

$$V_{12}(\vec{q}) = \int \mathrm{d}\vec{r}\, \mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{r}} V(\vec{r}_1 s_1, \vec{r}_2 s_2) = \mu^2 \sum_{M=-2}^{+2} h_{-M} N_{12}^{(M)} Y_{2,-M}(\widehat{q}), \tag{2.7}$$

where h_{-M} are constants given by

$$h_0 = (4\pi/3)\sqrt{\frac{16\pi}{5}}, \qquad h_{\pm 1} = \pm(4\pi/3)\sqrt{\frac{6\pi}{5}}, \qquad h_{\pm 2} = (4\pi/3)\sqrt{\frac{6\pi}{5}}. \tag{2.8}$$

In this problem, $V_{12}(\vec{q})$ is independent of the magnitude of $\vec{q}$, but depends on the direction through the spherical harmonic $Y_{2,-M}(\theta_q, \phi_q)$. It is however spin dependent through the operators $N_{12}^{(M)}$. It can be shown directly that $V_{12}(\vec{q} = 0) = 0$.

Now let us consider the state of two particles with momenta $\hbar\vec{k}_1$ and $\hbar\vec{k}_2$ and z-component of spins σ_1 and σ_2, respectively, with the corresponding anti-symmetrized wavefunction. In the non-relativistic case, which we consider here, in such a state the kinetic energy of the two particles is $(\hbar^2/2m)\left(k_1^2 + k_2^2\right)$, whereas the first-order HF energy is given by

$$\begin{aligned}\langle V_{12}\rangle &= \left\langle \Phi_{\vec{k}_1\sigma_1,\vec{k}_2\sigma_2}(\vec{r}_1 s_1, \vec{r}_2 s_2) \middle| V(\vec{r}_1 s_1, \vec{r}_2 s_2) \middle| \Phi_{\vec{k}_1\sigma_1,\vec{k}_2\sigma_2}(\vec{r}_1 s_1, \vec{r}_2 s_2)\right\rangle \\ &= \frac{\mu^2}{V^2} \int \mathrm{d}\vec{r}\, \frac{1}{r^3} \left[\begin{array}{l} F^{(0)}(\widehat{r}) \langle \sigma_1(s_1)\sigma_2(s_2)| N_{12}^{(0)} |\sigma_1(s_1)\sigma_2(s_2)\rangle \\ \quad - \mathrm{e}^{-\mathrm{i}(\vec{k}_1-\vec{k}_2)\cdot\vec{r}} F^{(0)}(\widehat{r}) \langle \sigma_2(s_1)\sigma_1(s_2)| N_{12}^{(0)} |\sigma_1(s_1)\sigma_2(s_2)\rangle \end{array}\right].\end{aligned} \tag{2.9}$$

Here, the first term inside the bracket is the direct term and the second term is the exchange term. Note that in the first-order correction to the energy, only the $M = 0$ term in the interaction contributes; it does not change the z-component of the total spin S_z. The direct term ($\vec{q} = \vec{k}_1 - \vec{k}_2 = 0$) after angular integration goes to zero because $F^{(0)}(\widehat{r})$ is proportional to $Y_{20}(\widehat{r})$. In fact, the exchange term is related to the $\vec{q} = \vec{k}_1 - \vec{k}_2 \neq 0$ component of the FT of the part of the interaction with $M = 0$. Using the result (2.7), it can be explicitly written as

$$E_{12}^{\text{exch}} = -\frac{\mu^2}{V}\left(\frac{4\pi}{3}\right)\left(\frac{\sqrt{16\pi}}{\sqrt{5}}\right) Y_{20}(\widehat{q})\bar{N}_{12}^{(0)}(\sigma_1, \sigma_2), \tag{2.10}$$

where the spin matrix element is given by

$$\begin{aligned}\bar{N}_{12}^{(0)}(\sigma_1, \sigma_2) &= \langle \sigma_2(s_1)\sigma_1(s_2)|N_{12}^{(0)}|\sigma_1(s_1)\sigma_2(s_2)\rangle \\ &= \tfrac{1}{4}\delta_{\sigma_1,\sigma_2} - \tfrac{1}{4}\big(\delta_{\sigma_1,-1/2}\delta_{\sigma_2,1/2} + \delta_{\sigma_1,1/2}\delta_{\sigma_2,-1/2}\big).\end{aligned} \tag{2.11}$$

In equation (2.11), $\delta_{\sigma,\sigma'}$ is a Kronecker delta function. A close examination of (2.10) and (2.11) shows that the exchange contribution lowers the energy (negative overall) when the spins are parallel and the direction of the momentum transfer $\vec{q}$ is such that $\cos^2\theta_q > 1/3$, or when the spins are antiparallel and $\cos^2\theta_q < 1/3$. The maximum lowering of energy comes from the parallel spins with wave vectors such that $\theta_q = 0$ or π, with $\vec{k}_1 \neq \vec{k}_2$.

The many-body Hamiltonian for this system of N particles in volume V is given by

$$\begin{aligned}H = \sum_{\vec{k}\sigma} \varepsilon_{\vec{k}\sigma} c_{\vec{k}\sigma}^{+} c_{\vec{k}\sigma} + \frac{\mu^2}{2V} \sum_{\vec{k}_1\sigma_1\sigma_1'} \sum_{\vec{k}_2\sigma_2\sigma_2'} \sum_{\vec{q}} \langle \sigma_1'(s_1)\sigma_2'(s_2)|V_{12}(\vec{q})|\sigma_1(s_1)\sigma_2(s_2)\rangle \\ \times c_{\vec{k}_1+\vec{q}\sigma_1'}^{+} c_{\vec{k}_2-\vec{q}\sigma_2'}^{+} c_{\vec{k}_2\sigma_2} c_{\vec{k}_1\sigma_1}\end{aligned} \tag{2.12}$$

where $c_{\vec{k}\sigma}^{+}$ ($c_{\vec{k}\sigma}$) are the usual creation (destruction) operators for the particles, and $\varepsilon_{\vec{k}\sigma} = \hbar^2k^2/2m$. The FT of the interaction operator, defined in equation (2.7), gives a spin-dependent interaction depending only on the direction of the momentum transfer but not on its magnitude. In terms of the inter-particle separation

$$r_0 = (3V/4\pi N)^{1/3} = (3/4\pi n)^{1/3}, \tag{2.13}$$

we can scale different quantities in equation (2.12) in dimensionless form as

$$\vec{p} = r_0\vec{k}, \qquad \bar{V} = V/r_0^3, \qquad \vec{\mathcal{q}} = r_0\vec{q}, \tag{2.14}$$

so that the Hamiltonian (2.12) can be rewritten as

$$\begin{aligned}H = \frac{\hbar^2}{2mr_0^2} \sum_{\vec{p}\sigma} p^2 c_{\vec{p}\sigma}^{+} c_{\vec{p}\sigma} + \frac{\mu^2}{2\bar{V}r_0^3} \sum_{\vec{p}_1\sigma_1\sigma_1'} \sum_{\vec{p}_2\sigma_2\sigma_2'} \sum_{\vec{\mathcal{q}}} \langle \sigma_1'(s_1)\sigma_2'(s_2)|V_{12}(\widehat{Q})|\sigma_1(s_1)\sigma_2(s_2)\rangle \\ \times c_{\vec{p}_1+\vec{\mathcal{q}}\sigma_1'}^{+} c_{\vec{p}_2-\vec{\mathcal{q}}\sigma_2'}^{+} c_{\vec{p}_2\sigma_2} c_{\vec{p}_1\sigma_1}.\end{aligned} \tag{2.15}$$

Note that in the above expression, the quantities inside the summation signs are dimensionless. It implies that the interaction term is small compared to the kinetic energy for low densities, i.e. if $r_{\text{sm}} \equiv (\hbar^2 r_0/2m\mu^2) > 1$, but for high densities when $r_{\text{sm}} \ll 1$, it dominates the kinetic energy term. In other words, in terms of the magnetic radius $r_{\text{m}} = (2m\mu^2/\hbar^2)$, and the ratio $r_{\text{sm}} = r_0/r_{\text{m}}$, we can treat the interaction term as a perturbation to the non-interacting state only if $r_{\text{sm}} \gg 1$. Since we are not interested here in this limit, we will not use the usual perturbative approach, but examine the expectation value of the above Hamiltonian in a trial ground state using the variational approach. We will assume that in the variational ground state $|\Psi_0\rangle$ the single-particle occupation function $n_\sigma(\vec{k})$ is not necessarily the spherical Fermi function $f_0(k)$ of the non-interacting system, and treat it as a variational parameter. We also assume that there is no mass density wave; this drops the direct term.

3. The exchange energy and the ground state of the system

As indicated before, we are not interested here in the low-density perturbation approach in which V_{12} can be considered small compared to the kinetic energy, and where one can start with the non-interacting paramagnetic ground state $|0\rangle$. This is in contrast to the electron gas problem where in the high density limit the kinetic energy dominates the interaction energy which can therefore be treated perturbatively (after properly taking care of the $q = 0$ singularity of the Coulomb interaction). For the dipolar quantum fluid case we assume that the variational single-determinant ground state $|\Psi_0\rangle$ is described by a more general single-particle distribution function, similar to the expansion used in the Landau Fermi liquid theory [4],

$$n_\sigma(\vec{k}) = n_\sigma^{(0)}(k) + \Delta n_\sigma(\vec{k}) = n_\sigma^{(0)}(k) + \sum_{l\neq 0}\sum_{m=-l}^{l} \Delta n_\sigma^{lm}(k) Y_{lm}(\widehat{k}), \tag{3.1}$$

in which $n_\sigma^{(0)}(k)$ is the spherical part of the distribution function and $\Delta n_\sigma(\vec{k})$ is the angular part. This division is motivated by the fact that the interaction is non–central; the non-spherical part is a measure of the deformation of the spherical Fermi surface of the non-interacting system. Also note that the spherical part $n_\sigma^{(0)}(k)$ need not coincide with the non-interacting Fermi distribution function $f_0(k)$, which at $T = 0$ K equals the theta-function, i.e. it is equal to 1 for k below the common Fermi wave vector (in the paramagnetic state):

$$k_{F0} \equiv (3\pi^2 n)^{1/3} \tag{3.2}$$

and vanishes for k above. The expectation value of the Hamiltonian in the state (3.1) is

$$\langle\Psi_0|H|\Psi_0\rangle \equiv E = E_{\text{kin}} + E_{\text{exch}}, \tag{3.3}$$

where the kinetic energy contribution is given by

$$E_{\text{kin}} = \sum_{\vec{k}\sigma} (\hbar^2 k^2/2m) n_\sigma(\vec{k}), \tag{3.4}$$

and the exchange energy is given by

$$E_{\text{exch}} = -\frac{\mu^2}{2V} h_0 \sum_{\vec{k}\vec{q}} \sum_{\sigma_1\sigma_2} n_{\sigma_1}(\vec{k}+\vec{q}) n_{\sigma_2}(\vec{k}) Y_{20}(\widehat{q}) \bar{N}_{12}^{(0)}(\sigma_1, \sigma_2), \tag{3.5}$$

where the constant $h_0 = (4\pi/3)\left(\sqrt{16\pi/5}\right)$ and the spin matrix elements $\bar{N}_{12}^{(0)}(\sigma_1, \sigma_2)$ are given by equation (2.11).

If we substitute the form (3.1) of the distribution function in the above expressions for E_{kin} and E_{exch}, we observe that

$$\sum_{\vec{k}} \Delta n_\sigma(\vec{k}) = 0, \tag{3.6}$$

and

$$\sum_{\vec{k}} \sum_{\vec{q}} n_{\sigma_1}^{(0)}(|\vec{k}+\vec{q}|) n_{\sigma_2}^{(0)}(k) Y_{20}(\widehat{q}) = 0, \tag{3.7}$$

because $\Delta n_\sigma(\vec{k})$ does not contain any $l = 0$ component and the integration over $\vec{k}$ in equation (3.7) gives a function which depends only on the magnitude of $\vec{q}$. We thus find

$$E_{\text{kin}} = \sum_{\vec{k}\sigma} \left(\frac{\hbar^2 k^2}{2m}\right) n_\sigma^{(0)}(k) \tag{3.8}$$

and

$$E_{\mathrm{exch}} = -\frac{\mu^2}{2V} h_0 I_{\mathrm{exch}}, \tag{3.9}$$

where

$$I_{\mathrm{exch}} = \sum_{\vec{k},\vec{q}} \sum_{\sigma_1,\sigma_2} \bar{N}_{12}^{(0)}(\sigma_1,\sigma_2) Y_{20}(\widehat{q}) \times \left\lfloor n_{\sigma_1}^{(0)}(|\vec{k}+\vec{q}|)\Delta n_{\sigma_2}(\vec{k}) + n_{\sigma_2}^{(0)}(|\vec{k}+\vec{q}|)\Delta n_{\sigma_1}(\vec{k}) + \Delta n_{\sigma_1}(\vec{k}+\vec{q})\Delta n_{\sigma_2}(\vec{k})\right\rfloor. \tag{3.10}$$

The first two terms of equation (3.10), linear in the deformation, contribute only if the deformation $\Delta n_\sigma(\vec{k})$ has $l = 2, 4, 6, \ldots$ components, because the decomposition [5] of the spherical function $n_\sigma^{(0)}(|\vec{k}+\vec{q}|)$ contains an infinite sum of products of $Y_{lm}(\widehat{k})Y_{lm}^*(\widehat{q})$:

$$\begin{aligned} n_\sigma^{(0)}(|\vec{k}+\vec{q}|) &= \sum_{l=0}^{\infty} \tilde{n}_{\sigma l}^{(0)}(k,q) P_l(\cos\theta_{kq}) \\ &= \sum_{l=0}^{\infty} \tilde{n}_{\sigma l}^{(0)}(k,q) \left(\frac{4\pi}{2l+1}\right) \sum_m Y_{lm}(\widehat{q}) Y_{lm}^*(\widehat{k}), \end{aligned} \tag{3.11}$$

where

$$\tilde{n}_{\sigma l}^{(0)} = (2l+1)/2 \int_{-1}^{+1} \mathrm{d}(\cos\theta_{kq}) P_l(\cos\theta_{kq}) n_\sigma^{(0)}(|\vec{k}+\vec{q}|). \tag{3.12}$$

Here, $P_l(\cos\theta)$ is the Legendre function. For the $l = 1$ deformation, one has to go to the last term in equation (3.10), which is of the second order in deformation, to obtain any nonvanishing contribution.

In principle, to minimize the total energy one has to vary all possible parameters in the assumed $n_\sigma^{(0)}(k)$ and $\Delta n_\sigma(\vec{k})$. However, let us suppose that for small deformations we can deal with only the linear terms in equation (3.10), and assume further that we have only $l = 2$ deformation, the lowest l possible in this case:

$$\Delta n_\sigma(\vec{k}) = \Delta n_\sigma^{(20)}(k) Y_{20}(\widehat{k}). \tag{3.13}$$

Because of symmetry, the terms with $Y_{2m}(\widehat{k})$, $m \neq 0$, do not contribute to linear terms. This simplification leads to

$$I_{\mathrm{exch}} \approx \sum_{\vec{k}\vec{q}} \sum_{\sigma_1\sigma_2} \bar{N}_{12}^{(0)}(\sigma_1,\sigma_2) Y_{20}(\widehat{q}) n_{\sigma_1}^{(0)}(|\vec{k}+\vec{q}|) \Delta n_{\sigma_2}^{(20)}(k) Y_{20}(\widehat{k}) + (\sigma_1 \leftrightarrow \sigma_2). \tag{3.14}$$

This expression can also be rewritten in the form ($\vec{k}_{12} = \vec{k}_1 - \vec{k}_2$)

$$I_{\mathrm{exch}} = \sum_{\vec{k}_1\vec{k}_2} \sum_{\sigma_1\sigma_2} \bar{N}_{12}^{(0)}(\sigma_1,\sigma_2) n_{\sigma_1}^{(0)}(k_1) \Delta n_{\sigma_2}^{(20)}(k_2) Y_{20}(\widehat{k}_2) Y_{20}(\widehat{k}_{12}) + (\sigma_1 \leftrightarrow \sigma_2). \tag{3.15}$$

One may use either of the above forms to proceed further. If one uses the form (3.14), one needs to expand $n_\sigma^{(0)}(|\vec{k}+\vec{q}|)$ in spherical harmonics $Y_{lm}(\widehat{k})Y_{lm}^*(\widehat{q})$ as done already in equations (3.11) and (3.12), whereas in the form (3.15) one has to express $Y_{20}(\widehat{k}_{12})$ as products of spherical harmonics $Y_{lm}(\widehat{k}_1)Y_{l'm'}^*(\hat{k}_2)$. In the latter case, l and l' are not necessarily the same in the sum of the products. Since both the expressions must give the same result, we use the form (3.14) and expand $n_\sigma^{(0)}(|\vec{k}+\vec{q}|)$ as in equations (3.11) and (3.12). After some algebra, including angular integrations and spin summations, we finally obtain

$$I_{\mathrm{exch}} = \frac{1}{(4\pi)^2} \sum_{\vec{k},\vec{q}} \frac{1}{2} \left[\left\{\tilde{n}_{\uparrow 2}^{(0)}(k,q) - \tilde{n}_{\downarrow 2}^{(0)}(k,q)\right\}\left\{\Delta n_\uparrow^{(20)}(k) - \Delta n_\downarrow^{(20)}(k)\right\}\right]. \tag{3.16}$$

Now, we must choose proper forms for $n_\sigma^{(0)}(k)$ and $\Delta n_\sigma^{(20)}(k)$, with the constraint that the total occupation number $n_\sigma^{(0)}(k) + \Delta n_\sigma^{(20)}(k) Y_{20}(\widehat{k})$ is non-negative and does not exceed 1 for any $\vec{k}$. We know that in the absence of kinetic energy, the exchange energy itself is minimum if all the spins are parallel and different particles move with different momenta in the positive z-direction or the negative z-direction. In view of this let us assume that there is no occupation of down-spin states. In this case

$$\tilde{n}_{\downarrow 2}^{(0)}(k,q) = 0, \qquad \Delta n_\downarrow^{(20)}(k) = 0 \tag{3.17}$$

and we have all spins up (fully polarized state). Thus, the exchange energy becomes

$$I_{\text{exch}} = \frac{1}{(4\pi)^2} \sum_{\vec{k},\vec{q}} \frac{1}{2} \tilde{n}_{\uparrow 2}^{(0)}(k,q) \Delta n_\uparrow^{(20)}(k). \tag{3.18}$$

For definiteness, let us try the following simple ansatz for $n_\uparrow^{(0)}(k)$ and $\Delta n_\uparrow^{(20)}(k)$:

$$n_\uparrow^{(0)}(k) = (1 - |\beta|) f_0(k); \qquad \Delta n_\uparrow^{(20)}(k) = \frac{\beta}{2}\sqrt{\frac{16\pi}{5}} f_0(k), \tag{3.19a}$$

so that

$$n_\uparrow(\vec{k}) = (1 - |\beta|) f_0(k) + \frac{\beta}{2}(3\cos^2\theta_k - 1) \tag{3.19b}$$

with $-\frac{1}{2} < \beta < \frac{2}{3}$. Here $f_0(k)$ is the Fermi distribution function. This range of values for β, which can be taken to be either negative or positive, ensures that the total $n_\downarrow(k)$ is non-negative and not larger than 1 for any value of θ_k. Since we have neglected the second-order terms in deformation, we will of course like β to be small. The substitution of (3.19) into (3.18) then leads to

$$I_{\text{exch}} = \beta(1 - |\beta|) \frac{1}{4}\sqrt{\frac{16\pi}{5}} \left(\frac{1}{4\pi}\right)^2 \sum_{\vec{k}\vec{q}} f_{02}(k,q) f_0(q) \tag{3.20}$$

where

$$\begin{aligned} f_{02}(k \cdot q) &= \frac{5}{2} \int_{-1}^{+1} \mathrm{d}z\, f_0\left(\sqrt{(k^2 + q^2 + 2kqz)}\right) P_2(z); \\ P_2(z) &= \frac{1}{2}(3z^2 - 1). \end{aligned} \tag{3.21}$$

Note that at $T = 0$ K, the magnitudes of $\vec{k}$ and $|\vec{k} + \vec{q}|$ have to be less than the new Fermi wave vector:

$$k_{F\uparrow} = k_{F0} \left(\frac{2}{(1 - |\beta|)}\right)^{1/3}; \qquad k_{F0} = \left(\frac{3\pi^2 N}{V}\right)^{1/3}. \tag{3.22}$$

If we scale q and k in the dimensionless forms, $x = q/2k_{F\uparrow}$; $y = k/k_{F\uparrow}$, the summations over $\vec{k}$ and $\vec{q}$ in equation (3.20) can be simplified (for $T = 0$ K) to the form

$$(1/(4\pi)^2) \sum_{\vec{k}} \sum_{q} f_{02}(k,q) f_0(k) = \left(\frac{5}{2\pi}\right) N^2 J/(1 - |\beta|)^2, \tag{3.23}$$

where

$$J = \frac{9}{4\pi} \int_0^1 x^2\, \mathrm{d}x \int_0^1 y^2\, \mathrm{d}y\, f_{02}(x,y) \tag{3.24}$$

and

$$f_{02}(x, y) = \frac{5}{2}\int_{-1}^{+1} \mathrm{d}z\, f_0\left(\sqrt{(x^2 + y^2 + 2xyz)}\right) P_2(z). \tag{3.25}$$

Now, in our simplified case, equations (3.9), (3.20) and (3.23) directly give

$$E_{\mathrm{ex}}/N = -\frac{\mu^2}{r_0^3}\frac{J\beta}{(1 - |\beta|)}. \tag{3.26}$$

The dimensionless constant J given by the integral (3.24) is expected to be positive. A numerical calculation of the integral in fact gives its value to be 0.011 56. However, in reality it does not matter whether it is positive or negative. One can always choose the sign of β such that βJ is positive. Thus, we can replace βJ in the above equation by $|\beta||J|$. If J is positive, as in our case, β has to be positive. The assumed form (3.19*b*) for the occupation function leads to the expression for kinetic energy per particle as

$$E_{\mathrm{kin}}/N = \frac{3}{5}\frac{\hbar^2 k_{F0}^2}{2m}\left(\frac{2}{1 - |\beta|}\right)^{2/3} = \frac{2.21}{r_0^2}\left(\frac{\hbar^2}{2m}\right)\left(\frac{2}{1 - |\beta|}\right)^{2/3}, \tag{3.27}$$

where $(3/5)\left(\hbar^2 k_{F0}^2/2m\right) = \left(2.21/r_0^2\right)(\hbar^2/2m) \equiv E_0/N$ is the kinetic energy per particle for the non-interacting gas in the paramagnetic state. The kinetic energy has increased now because of the larger up-spin Fermi sphere which has to accommodate all the N particles.

The total energy per particle becomes

$$E/N = \frac{2.21}{r_0^2}\left(\frac{\hbar^2}{2m}\right)\left[\left(\frac{2}{(1 - |\beta|)}\right)^{2/3} - \frac{1}{r_{\mathrm{sm}}^*}\left(\frac{|\beta|}{(1 - |\beta|)}\right)\right], \tag{3.28}$$

with the dimensionless density parameter

$$\frac{1}{r_{\mathrm{sm}}^*} = \frac{r_{\mathrm{m}}}{r_0}\frac{|J|}{2.21} \equiv \frac{r_{\mathrm{m}}^*}{r_0} = \frac{2m\mu^2}{\hbar^2}\left(\frac{|J|}{2.21}\right)\frac{1}{r_0} \approx \frac{1}{r_{\mathrm{sm}}}(1/191). \tag{3.29}$$

In our case, since we found J to be positive, we must have β also positive, with $0 < \beta < 2/3$, so that the occupation number is non-negative and does not exceed 1. Note that for our dipolar Hamiltonian, E_0/N is also the variational ground state energy per particle if the trial ground state is taken to be the non-interacting paramagnetic state $|0\rangle$:

$$\langle 0|H|0\rangle/N = \left(2.21/r_0^2\right)(\hbar^2/2m) \equiv E_0/N. \tag{3.30}$$

The energy (3.28) becomes less than the energy per particle, E_0/N, of the non-interacting paramagnetic ground state, for r_0 such that

$$r_0 < r_{\mathrm{m}}^*\frac{\beta}{(1 - \beta)}\frac{1}{[\{2/(1 - \beta)\}^{2/3} - 1]}, \tag{3.31}$$

where $r_{\mathrm{m}}^* = r_{\mathrm{m}}J/(2.21) \approx r_{\mathrm{m}}/191$. In figure 1, we have plotted the ratio of the ground state energies E/E_0 for different values of the deformation parameter β in the allowed range of 0 to 2/3, as a function of $r_{\mathrm{sm}}^* = r_0/r_{\mathrm{m}}^*$. For a particle with mass m and magnetic moment of an electron, $r_{\mathrm{m}} \cong 10^{-13}$ cm, relation (3.31) implies that the transition will take place for $r_0 < 10^{-13}$ cm and density greater than 10^{+39} cm^3. For a particle with heavier mass but the same magnetic moment, the requirement is less severe, whereas for particles such as neutrinos with smaller mass (by a factor in the range of 10^{-6}–10^{-9} or so [6]) and much smaller magnetic moment (by a factor of 10^{-10} or more [7]), the requirement on density for this transition is extremely severe.

If one increases the density beyond the above transition, eventually the total energy becomes negative (see figure 1) for

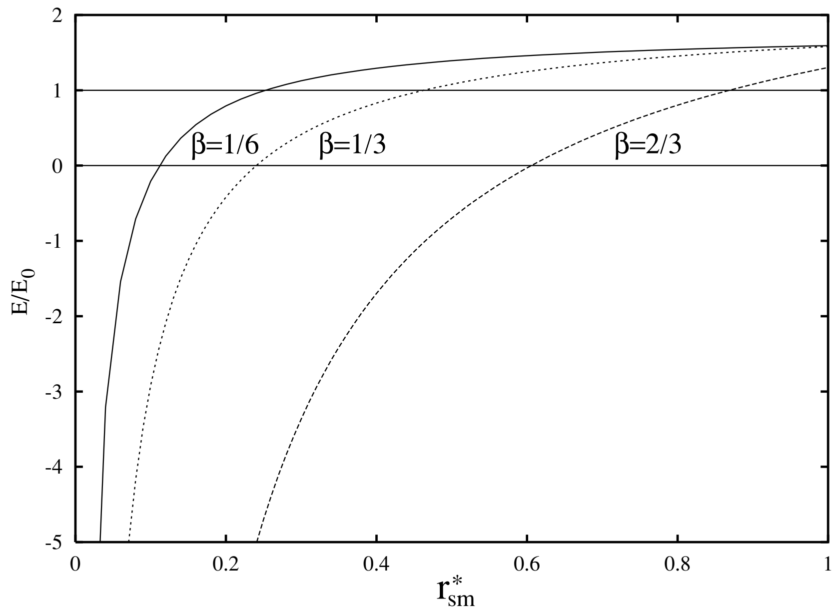

Figure 1. Comparison of the ground state energy E of the variational polarized ferromagnetic state having a deformed Fermi surface with the ground state energy E_0 of the non-interacting paramagnetic state. The ratio is plotted as a function of density parameter $r^*_{\mathrm{sm}} = r_0/r^*_{\mathrm{m}} \approx (3/4\pi n)^{1/3}/[(2m\mu^2/\hbar^2)\times 191]$ for different values of the deformation parameter β in the allowed range of 0 to 2/3.

$$r_{0,\mathrm{critical}} = r^*_{\mathrm{m}}\frac{\beta}{(1-\beta)}\left(\frac{1-\beta}{2}\right)^{2/3}; \qquad 0 < \beta < 2/3, \tag{3.32}$$

and the system will collapse to the infinite density state unless some other particle–particle interaction at very short distances can arrest this collapse.

4. Discussion of results

By choosing a very simple form for the single-particle occupation function $n_\sigma(\vec{k})$ for the new variational ground state $|\Psi_0\rangle$, we have shown that for a system of neutral fermions such as neutrinos interacting via magnetic dipole interaction, the total energy can indeed be lowered, compared to the case of the non-interacting ground state $|0\rangle$, due to the negative exchange energy contribution, when the density is sufficiently high so that the inter-particle distance $r_0 < r_{\mathrm{m}} = 2m\mu^2/\hbar^2$. As the density is increased further, for $r_0 < r_{0,\mathrm{critical}}$, the system will collapse to an infinite density state. In this exercise, we have used a simple variational form for the distribution function given by equations (3.17) and (3.19*b*), in which we have assumed a complete ($\delta = 1$) spin-polarized ferromagnetic state. It is possible to get a lowering of energy even if we assume that in addition to spin-up occupation and deformation in its occupation function, there is spin-down occupation $\left(n^{(0)}_\downarrow(\vec{k}) \neq 0\right)$, with no deformation ($\Delta n_\downarrow(\vec{k}) = 0$). In that case, there may be a range of polarization parameter $\delta < 1$, defined by $n_\uparrow - n_\downarrow = n\delta$, for which the system has a lower energy with a minimum as a function of δ for sufficiently high density. We plan to consider this problem in a later investigation.

Also, in this paper we have restricted our variational occupation function to the $l = 2$ deformations, and have neglected the term in the exchange energy which is of the second-order deformation. If we include the second-order term, it will be possible to consider the $l = 1$ deformations and see whether this can give a nonvanishing contribution and lead to a lower energy than that calculated here. This will also be taken up in a later publication. The main motivation of our study in this paper was to show that the dipolar exchange energy in a quantum Fermi liquid (QFL) can indeed lower the energy in a spin-polarized state with a deformed Fermi sphere, and above a critical density the system will collapse to an infinite density sate. We have not found any earlier calculation of the exchange energy in this case which could throw light on the nature of the ground state of chargeless fermions with only magnetic dipole interaction. In [1], the existence of a fairly large attractive exchange energy is assumed in a phenomenological model equivalent to the phenomenological Stoner-like model for ferromagnetic transition involving itinerant electrons in metals. However, one cannot use directly such a model here, because the form and the nature of the exchange interaction arising from the magnetic dipole interaction are completely different than that coming from the usual Coulomb interaction. In completely different contexts, there are earlier discussions of the effect of additional dipolar interaction [8, 9] for particles on a lattice or in classical fluids while considering magnetic or other phase transitions in condensed matter. In those studies, other molecular interactions are always present in the system, in addition to the magnetic dipole (or the electric dipole or the elastic dipole) interaction. The problem considered by us is completely different than those, because here we have the full three-dimensional positive kinetic energy contribution of the QFL, as the quantum dipoles are allowed to move, and there is no other interaction present in the system except for the magnetic dipole interaction between the particles.

One important question which we must discuss at this point is whether higher order terms in the interaction, similar to the Goldstone diagrams [2] of the perturbation theory, would change the validity of our main results. This is particularly relevant because in our case the interaction is independent of the magnitude of the momentum transfer, depending only on its direction, and we are dealing with the high density region. In this connection, we must, however, note that we are not using the straightforward perturbation theory on the non-interacting ground state $|0\rangle$. We have used a variational ground state $|\Psi_0\rangle$ to calculate its energy $E = \langle\Psi_0|H|\Psi_0\rangle$. The variational principle ensures that the true ground state energy must always be lower than the energy E calculated here. In other words, the existence of a deformed magnetic state with lower energy compared to the uniform paramagnetic state and infinite density collapse above a critical density will survive in an exact calculation. The critical density itself would be lower than that obtained here. We propose to investigate this further in a later work.

As we have presented our results in section 3, for neutrinos with a much smaller mass and a very tiny magnetic moment compared to electrons, the critical density for the transition will be exceedingly high. In cosmological context, very close to the big-bang one has very high densities, but one also has very high temperatures. Our results of the zero-temperature ground state calculation cannot be applied directly to such a situation. We must extend our work not only to finite temperatures but also to relativistic single-particle energies. Here, we will prefer not to speculate about the impact of our work on neutrino cosmology, and wait for the results of such a calculation to be taken up in future.

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